The Brézis-Browder Theorem revisited and properties of Fitzpatrick functions of order n

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May 22, 2009

Abstract

In this note, we study maximal monotonicity of linear relations (set-valued operators with linear graphs) on reflexive Banach spaces. We provide a new and simpler proof of a result due to Brézis-Browder which states that a monotone linear relation with closed graph is maximal monotone if and only if its adjoint is monotone. We also study Fitzpatrick functions and give an explicit formula for Fitzpatrick functions of order n for monotone symmetric linear relations.

2000 Mathematics Subject Classification:
Primary 47A06, 47H05; Secondary 47A05, 47B65, 52A41, 90C25

Keywords: Adjoint, convex function, convex set, Fenchel conjugate, Fitzpatrick function, linear relation, maximal monotone operator, multifunction, monotone operator, set-valued operator, symmetric.

1 Introduction

Monotone operators play important roles in convex analysis and optimization [11, 17, 20, 21, 19, 27, 28]. In 1978, Brézis-Browder gave some characterizations of a monotone operator with closed graph ([10, Theorem 2]). The Brézis-Browder Theorem states that a monotone linear relation with closed graph is maximal monotone if and only if its adjoint is monotone if and only if its adjoint is maximal monotone, which gives the connection between the monotonicity of a linear relation and that of its adjoint. Now we give a new and simpler proof of the hard part of the Brézis-Browder Theorem (Theorem 2.5): a monotone linear relation with closed graph is maximal monotone if its adjoint is monotone.

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We suppose throughout this note that X is a real reflexive Banach space with norm $\|\cdot\|$, that X^* is its continuous dual space with norm $\|\cdot\|_*$, and dual product $\langle\cdot,\cdot\rangle$. We now introduce some notation. Let $A\colon X \rightrightarrows X^*$ be a *set-valued operator* or *multifunction* whose graph is defined by

$$\operatorname{gra} A := \big\{(x,x^*) \in X \times X^* \mid x^* \in Ax\big\}.$$

The *inverse operator* of A, $A^{-1}\colon X^* \rightrightarrows X$, is given by $\operatorname{gra} A^{-1} := \big\{(x^*,x) \in X^* \times X \mid x^* \in Ax\big\}$; the *domain* of A is $\operatorname{dom} A := \big\{x \in X \mid Ax \neq \varnothing\big\}$.

If Z is a real reflexive Banach space with dual Z^* and a set $S \subseteq Z$, we denote $S^\perp$ by $S^\perp := \big\{z^* \in Z^* \mid \langle z^*, s\rangle = 0, \quad \forall s \in S\big\}$. Then the *adjoint* of A, denoted by A^*, is defined by

$$\operatorname{gra} A^* := \big\{(x,x^*) \in X \times X^* \mid (x^*,-x) \in (\operatorname{gra} A)^\perp\big\}.$$

Note that A is said to be a *linear relation* if $\operatorname{gra} A$ is a linear subspace of $X \times X^*$. (See [14] for further information on linear relations.) Recall that A is said to be *monotone* if for all $(x,x^*),(y,y^*) \in \operatorname{gra} A$ we have

$$\langle x-y, x^*-y^*\rangle \geq 0,$$

and A is *maximal monotone* if A is monotone and A has no proper monotone extension. We say $(x,x^*) \in X \times X^*$ is *monotonically related* to $\operatorname{gra} A$ if (for every $(y,y^*) \in \operatorname{gra} A$) $\langle x-y, x^*-y^*\rangle \geq 0$. Recently linear relations have been become an interesting object and comprehensively studied in Monotone Operator Theory: see [1, 2, 3, 6, 7, 8, 18, 24, 25, 26]. We can now precisely describe the Brézis-Browder Theorem. Let A be a monotone linear relation with closed graph. Then

$$\begin{aligned} A \text{ is maximal monotone} &\Leftrightarrow A^* \ \text{ is maximal monotone}\\ &\Leftrightarrow A^* \ \text{ is monotone.}\end{aligned}$$

Our goal of this paper is to give a simpler proof of Brézis-Browder Theorem and to derive more properties of Fitzpatrick functions of order n. The paper is organized as follows. The first main result (Theorem 2.5) is proved in Section 2 providing a new and simpler proof of the Brézis-Browder Theorem. In Section 3, some explicit formula for Fitzpatrick functions are given. Recently, *Fitzpatrick functions of order n* [1] have turned out to be a useful tool in the study of n-cyclic monotonicity (see [1, 4, 3]). Theorem 3.10 gives an explicit formula for Fitzpatrick functions of order n associated with symmetric linear relations, which generalizes and simplifies [1, Example 4.4] and [3, Example 6.4].

Our notation is standard. The notation $A\colon X \to X^*$ means that A is a *single-valued* mapping (with full domain) from X to X^*. Given a subset C of X, $\overline{C}$ is the closure of C. The *indicator function* $\iota_C : X \to \left]-\infty,+\infty\right]$ of C is defined by

$$x \mapsto \begin{cases} 0, & \text{if } x \in C;\\ +\infty, & \text{otherwise.}\end{cases} \tag{1}$$

For a function $f\colon X \to \left]-\infty,+\infty\right]$, $\operatorname{dom} f = \{x \in X \mid f(x) < +\infty\}$ and $f^*\colon X^* \to [-\infty,+\infty] : x^* \mapsto \sup_{x\in X}(\langle x,x^*\rangle - f(x))$ is the *Fenchel conjugate* of f. Recall that f

is said to be proper if $\operatorname{dom} f \neq \varnothing$. If f is convex, $\partial f\colon X \rightrightarrows X^*\colon x \mapsto \big\{x^* \in X^* \mid (\forall y \in X)\ \langle y - x, x^*\rangle + f(x) \leq f(y)\big\}$ is the *subdifferential operator* of f. Denote J by the duality map, i.e., the subdifferential of the function $\frac{1}{2}\|\cdot\|^2$, by [17, Example 2.26],

$$Jx := \{x^* \in X^* \mid \langle x^*, x\rangle = \|x^*\|_* \cdot \|x\|, \text{ with } \|x^*\|_* = \|x\|\}.$$

2 New proof of the Brézis-Browder Theorem

Fact 2.1 (Simons) (See [21, Lemma 19.7 and Section 22].) *Let $A : X \rightrightarrows X^*$ be a monotone linear relation such that* $\operatorname{gra} A \neq \varnothing$. *Then the function*

$$g\colon X \times X^* \to \left]-\infty, +\infty\right] : (x, x^*) \mapsto \langle x, x^*\rangle + \iota_{\operatorname{gra} A}(x, x^*) \tag{2}$$

is proper and convex.

Fact 2.2 (Simons-Zălinescu) (See [22, Theorem 1.2].) *Let $A\colon X \rightrightarrows X^*$ be monotone. Then A is maximal monotone if and only if*

$$\operatorname{gra} A + \operatorname{gra}(-J) = X \times X^*.$$

Remark 2.3 When J and J^{-1} are single-valued, Fact 2.2 yields Rockafellar's characterization of maximal monotonicity of A. See [22, Theorem 1.3] and [21, Theorem 29.5 and Remark 29.7].

Now we state the Brézis-Browder Theorem.

Theorem 2.4 (Brézis-Browder) (See [10, Theorem 2].) *Let $A\colon X \rightrightarrows X^*$ be a linear relation with closed graph. Then the following statements are equivalent.*

(i) *A is maximal monotone.*

(ii) *A^* is maximal monotone.*

(iii) *A^* is monotone.*

Proof. (i)⇒(iii): Suppose to the contrary that A^* is not monotone. Then there exists $(x_0, x_0^*) \in \operatorname{gra} A^*$ such that $\langle x_0, x_0^*\rangle < 0$. Now we have

$$\begin{aligned}\langle -x_0 - y, x_0^* - y^*\rangle &= \langle -x_0, x_0^*\rangle + \langle y, y^*\rangle + \langle x_0, y^*\rangle + \langle -y, x_0^*\rangle \\ &= \langle -x_0, x_0^*\rangle + \langle y, y^*\rangle > 0, \quad \forall (y, y^*) \in \operatorname{gra} A.\end{aligned} \tag{3}$$

Thus, $(-x_0, x_0^*)$ is monotonically related to $\operatorname{gra} A$. By maximal monotonicity of A, $(-x_0, x_0^*) \in \operatorname{gra} A$. Then $\langle -x_0 - (-x_0), x_0^* - x_0^*\rangle = 0$, which contradicts (3). Hence A^* is monotone.

The hard parter is to show (iii)⇒(i). See Theorem 2.5 below.

(i)⇔(ii): Apply directly (iii)⇔(i) by using $A^{**} = A$ (since $\operatorname{gra} A$ is closed). ■

In Theorem 2.5, we provide a new and simpler proof to show the hard part (iii)⇒(i) in Theorem 2.4. The proof was inspired by that of [28, Theorem 32.L].

Theorem 2.5 *Let $A\colon X \rightrightarrows X^*$ be a monotone linear relation with closed graph. Suppose A^* is monotone. Then A is maximal monotone.*

Proof. We show that $X \times X^* \subseteq \operatorname{gra} A + \operatorname{gra}(-J)$. Let $(x, x^*) \in X \times X^*$ and we define $g : X \times X^* \to \left]-\infty, +\infty\right]$ by

$$(y, y^*) \mapsto \tfrac{1}{2}\|y^*\|_*^2 + \tfrac{1}{2}\|y\|^2 + \langle y^*, y\rangle + \iota_{\operatorname{gra} A}(y - x, y^* - x^*).$$

Since $\operatorname{gra} A$ is closed, g is lower semicontinuous on $X \times X^*$. By Fact 2.1, g is convex and coercive. Here g has minimizer. Suppose that (z, z^*) is a minimizer of g. Then $(z - x, z^* - x^*) \in \operatorname{gra} A$, that is,

$$(x, x^*) \in \operatorname{gra} A + (z, z^*). \tag{4}$$

On the other hand, since (z, z^*) is a minimizer of g, $(0, 0) \in \partial g(z, z^*)$. By a result of Rockafellar (see [13, Theorem 2.9.8]), there exist $(z_0^*, z_0) \in \partial(\iota_{\operatorname{gra} A}(\cdot - x, \cdot - x^*))(z, z^*) = \partial\iota_{\operatorname{gra} A}(z - x, z^* - x^*) = (\operatorname{gra} A)^{\perp}$, and $(v, v^*) \in X \times X^*$ with $v^* \in Jz$, $z^* \in Jv$ such that

$$(0, 0) = (z^*, z) + (v^*, v) + (z_0^*, z_0).$$

Then

$$\big(-(z + v), z^* + v^*\big) \in \operatorname{gra} A^*.$$

Since A^* is monotone,

$$\langle z^* + v^*, z + v\rangle = \langle z^*, z\rangle + \langle z^*, v\rangle + \langle v^*, z\rangle + \langle v^*, v\rangle \le 0. \tag{5}$$

Note that since $\langle z^*, v\rangle = \|z^*\|_*^2 = \|v\|^2$, $\langle v^*, z\rangle = \|v^*\|_*^2 = \|z\|^2$, by (5), we have

$$\tfrac{1}{2}\|z\|^2 + \tfrac{1}{2}\|z^*\|_*^2 + \langle z^*, z\rangle + \tfrac{1}{2}\|v^*\|_*^2 + \tfrac{1}{2}\|v\|^2 + \langle v, v^*\rangle \le 0.$$

Hence $z^* \in -Jz$. By (4), $(x, x^*) \in \operatorname{gra} A + \operatorname{gra}(-J)$. Thus, $X \times X^* \subseteq \operatorname{gra}(-J) + \operatorname{gra} A$. By Fact 2.2, A is maximal monotone. ■

3 Fitzpatrick functions and Fitzpatrick functions of order n

Now we introduce some properties of monotone linear relations.

Fact 3.1 (See [7].) *Assume that $A : X \rightrightarrows X^*$ is a monotone linear relation. Then the following hold.*

(i) *The function* $\operatorname{dom} A \to \mathbb{R} : y \mapsto \langle y, Ay \rangle$ *is convex.*

(ii) $\operatorname{dom} A \subseteq (A0)^{\perp}$. *For every* $x \in (A0)^{\perp}$, *the function* $\operatorname{dom} A \to \mathbb{R} : y \mapsto \langle x, Ay \rangle$ *is linear.*

Proof. (i): See [7, Proposition 2.3]. (ii): See [7, Proposition 2.2(i)(iii)]. ■

Definition 3.2 *Suppose* $A : X \rightrightarrows X^*$ *is a monotone linear relation. We say* A *is* symmetric *if* $\langle Ax, y \rangle = \langle Ay, x \rangle, \quad \forall x, y \in \operatorname{dom} A$.

For a monotone linear relation $A\colon X \rightrightarrows X^*$ it will be convenient to define (as in, e.g., [3])

$$q_A\colon X \to \mathbb{R}\colon x \mapsto \begin{cases} \frac{1}{2}\langle x, Ax \rangle, & \text{if } x \in \operatorname{dom} A; \\ \infty, & \text{otherwise.} \end{cases} \tag{6}$$

By Fact 3.1(i), q_A is at most single-valued and convex.

The following generalizes a result of Phelps-Simons (see [18, Theorem 5.1]) from symmetric monotone linear operators to symmetric monotone linear relations. We write $\overline{f}$ for the lower semicontinuous hull of f.

Proposition 3.3 *Let* $A : X \rightrightarrows X^*$ *be a monotone symmetric linear relation. Then*

(i) q_A *is convex, and* $\overline{q_A} + \iota_{\operatorname{dom} A} = q_A$.

(ii) $\operatorname{gra} A \subseteq \operatorname{gra} \partial \overline{q_A}$. *If* A *is maximal monotone, then* $A = \partial \overline{q_A}$.

Proof. Let $x \in \operatorname{dom} A$.

(i): Since A is monotone, q_A is convex. Let $y \in \operatorname{dom} A$. Since A is monotone, by Fact 3.1(ii),

$$0 \leq \tfrac{1}{2}\langle Ax - Ay, x - y \rangle = \tfrac{1}{2}\langle Ay, y \rangle + \tfrac{1}{2}\langle Ax, x \rangle - \langle Ax, y \rangle, \tag{7}$$

we have $q_A(y) \geq \langle Ax, y \rangle - q_A(x)$. Take lower semicontinuous hull and then deduce that $\overline{q_A}(y) \geq \langle Ax, y \rangle - q_A(x)$. For $y = x$, we have $\overline{q_A}(x) \geq q_A(x)$. On the other hand, $\overline{q_A}(x) \leq q_A(x)$. Altogether, $\overline{q_A}(x) = q_A(x)$. Thus (i) holds.

(ii): Let $y \in \operatorname{dom} A$. By (7) and (i),

$$q_A(y) \geq q_A(x) + \langle Ax, y - x \rangle = \overline{q_A}(x) + \langle Ax, y - x \rangle. \tag{8}$$

Since $\operatorname{dom} \overline{q_A} \subseteq \overline{\operatorname{dom} q_A} = \overline{\operatorname{dom} A}$, by (8), $\overline{q_A}(z) \geq \overline{q_A}(x) + \langle Ax, z - x \rangle, \quad \forall z \in \operatorname{dom} \overline{q_A}$. Hence $Ax \subseteq \partial \overline{q_A}(x)$. If A is maximal monotone, $A = \partial \overline{q_A}$. Thus (ii) holds. ■

Definition 3.4 *Let* $A\colon X \rightrightarrows X^*$. *The* Fitzpatrick function *of* A *is*

$$F_A\colon (x, x^*) \mapsto \sup_{(a,a^*) \in \operatorname{gra} A} \big(\langle x, a^* \rangle + \langle a, x^* \rangle - \langle a, a^* \rangle \big). \tag{9}$$

Definition 3.5 (Fitzpatrick family) *Let* $A\colon X \rightrightarrows X^*$ *be a maximal monotone operator. The associated* Fitzpatrick family $\mathcal{F}_A$ *consists of all functions* $F\colon X \times X^* \to \left]-\infty,+\infty\right]$ *that are lower semicontinuous and convex, and that satisfy* $F \geq \langle\cdot,\cdot\rangle$, *and* $F = \langle\cdot,\cdot\rangle$ *on* $\operatorname{gra} A$.

Following [16], it will be convenient to set $F^{\intercal}\colon X^* \times X\colon (x^*,x) \mapsto F(x,x^*)$, when $F\colon X \times X^* \to \left]-\infty,+\infty\right]$, and similarly for a function defined on $X^* \times X$.

Fact 3.6 (Fitzpatrick) (See [15, Theorem 3.10] or [12, Corollary 4.1].) *Let* $A\colon X \rightrightarrows X^*$ *be a maximal monotone operator. Then for every* $(x,x^*) \in X \times X^*$,

$$F_A(x,x^*) = \min\big\{F(x,x^*) \mid F \in \mathcal{F}_A\big\} \quad \textit{and} \quad F_A^{*\intercal}(x,x^*) = \max\big\{F(x,x^*) \mid F \in \mathcal{F}_A\big\}. \tag{10}$$

Proposition 3.7 *Let* $A\colon X \rightrightarrows X^*$ *be a maximal monotone and symmetric linear relation. Then*

$$F_A(x,x^*) = \tfrac{1}{2}\overline{q_A}(x) + \tfrac{1}{2}\langle x,x^*\rangle + \tfrac{1}{2}q_A^*(x^*), \quad \forall (x,x^*) \in X \times X^*.$$

Proof. Define function $k : X \times X^* \to \left]-\infty,+\infty\right]$ by

$$(z,z^*) \mapsto \tfrac{1}{2}\overline{q_A}(z) + \tfrac{1}{2}\langle z,z^*\rangle + \tfrac{1}{2}q_A^*(z^*).$$

Claim 1: $F_A = k$ on $\operatorname{dom} A \times X^*$.

Let $(x,x^*) \in X \times X^*$, and suppose that $x \in \operatorname{dom} A$. Then

$$\begin{aligned}
F_A(x,x^*) &= \sup_{(y,y^*)\in\operatorname{gra} A} \Big(\langle x,y^*\rangle + \langle y,x^*\rangle - \langle y,y^*\rangle\Big)\\
&= \sup_{y\in\operatorname{dom} A} \Big(\langle x,Ay\rangle + \langle y,x^*\rangle - 2q_A(y)\Big)\\
&= \tfrac{1}{2}\ q_A(x) + \sup_{y\in\operatorname{dom} A} \Big(\langle Ax,y\rangle + \langle y,x^*\rangle - \tfrac{1}{2}\ q_A(x) - 2q_A(y)\Big)\\
&= \tfrac{1}{2}\ q_A(x) + \tfrac{1}{2}\sup_{y\in\operatorname{dom} A} \Big(\langle Ax,2y\rangle + \langle 2y,x^*\rangle -\ q_A(x) - 4q_A(y)\Big)\\
&= \tfrac{1}{2}\ q_A(x) + \tfrac{1}{2}\sup_{z\in\operatorname{dom} A} \Big(\langle Ax,z\rangle + \langle z,x^*\rangle -\ q_A(x) - q_A(z)\Big)\\
&= \tfrac{1}{2}\ q_A(x) + \tfrac{1}{2}\sup_{z\in\operatorname{dom} A} \Big(\langle z,x^*\rangle -\ q_A(z-x)\Big)\\
&= \tfrac{1}{2}\ q_A(x) + \tfrac{1}{2}\langle x,x^*\rangle + \tfrac{1}{2}\sup_{z\in\operatorname{dom} A} \Big(\langle z-x,x^*\rangle -\ q_A(z-x)\Big)\\
&= \tfrac{1}{2}q_A(x) + \tfrac{1}{2}\langle x,x^*\rangle + \tfrac{1}{2}q_A^*(x^*)\\
&= k(x,x^*) \quad \text{(by Proposition 3.3(i))}.
\end{aligned}$$

Claim 2: k is convex and proper lower semicontinuous on $X \times X^*$.

Since F_A is convex, $\tfrac{1}{2}q_A + \tfrac{1}{2}\langle\cdot,\cdot\rangle + \tfrac{1}{2}q_A^*$ is convex on $\operatorname{dom} A \times X^*$. Now we show that k is convex. Let $\{(a,a^*),(b,b^*)\} \subseteq \operatorname{dom} k$, and $t \in \left]0,1\right[$. Then we have $\{a,b\} \subseteq \operatorname{dom}\overline{q_A} \subseteq \overline{\operatorname{dom} A}$. Thus, there

exist $(a_n),(b_n)$ in $\operatorname{dom} A$ such that $a_n \to a, b_n \to b$ with $q_A(a_n) \to \overline{q_A}(a), q_A(b_n) \to \overline{q_A}(b)$. Since $\frac{1}{2}q_A + \frac{1}{2}\langle\cdot,\cdot\rangle + \frac{1}{2}q_A^*$ is convex on $\operatorname{dom} A \times X^*$, we have

$$\begin{aligned}&\big(\tfrac{1}{2}q_A + \tfrac{1}{2}\langle\cdot,\cdot\rangle + \tfrac{1}{2}q_A^*\big)\big(ta_n + (1-t)b_n, ta^* + (1-t)b^*\big)\\ &\leq t\big(\tfrac{1}{2}q_A + \tfrac{1}{2}\langle\cdot,\cdot\rangle + \tfrac{1}{2}q_A^*\big)(a_n,a^*) + (1-t)\big(\tfrac{1}{2}q_A + \tfrac{1}{2}\langle\cdot,\cdot\rangle + \tfrac{1}{2}q_A^*\big)(b_n,b^*).\end{aligned} \tag{11}$$

Take $\liminf$ on both sides of (11) to see that

$$k\big(ta + (1-t)b, ta^* + (1-t)b^*\big) \leq tk(a,a^*) + (1-t)k(b,b^*).$$

Hence k is convex on $X \times X^*$. Thus, k is convex and proper lower semicontinuous.

Claim 3: $F_A = k$ on $X \times X^*$. To this end, we first observe that

$$\operatorname{dom} \partial k^* = \operatorname{gra} A^{-1}. \tag{12}$$

We have

$$\begin{aligned}&(w^*,w) \in \operatorname{dom} \partial k^* \Leftrightarrow (w^*,w) \in \operatorname{dom} \partial(2k)^* \Leftrightarrow (a,a^*) \in \partial(2k)^*(w^*,w), \quad \exists (a,a^*) \in X \times X^*\\ &\Leftrightarrow (w^*,w) \in \partial(2k)(a,a^*) \Leftrightarrow (w^*-a^*, w-a) \in \partial(\overline{q_A} \oplus q_A^*)(a,a^*), \quad \text{(by [13, Theorem 2.9.8])}\\ &\Leftrightarrow w^*-a^* \in \partial\overline{q_A}(a),\ w-a \in \partial q_A^*(a^*)\\ &\Leftrightarrow w^*-a^* \in \partial\overline{q_A}(a),\ a^* \in \partial\overline{q_A}(w-a)\\ &\Leftrightarrow w^*-a^* \in Aa,\ a^* \in A(w-a), \quad \text{(by Proposition 3.3(ii))}\\ &\Leftrightarrow (w,w^*) \in \operatorname{gra} A \Leftrightarrow (w^*,w) \in \operatorname{gra} A^{-1}.\end{aligned}$$

Next we observe that

$$k^{*\intercal}(z,z^*) = \langle z,z^*\rangle, \quad \forall (z,z^*) \in \operatorname{gra} A. \tag{13}$$

Since $k(z,z^*) \geq \langle z,z^*\rangle$ and

$$k(z,z^*) = \langle z,z^*\rangle \Leftrightarrow \overline{q_A}(z) + q_A^*(z^*) = \langle z,z^*\rangle \Leftrightarrow z^* \in \partial\overline{q_A}(z) = Az \quad \text{(by Proposition 3.3(ii))},$$

Fact 3.6 implies that $F_A \leq k \leq F_A^{*\intercal}$. Hence $F_A \leq k^{*\intercal} \leq F_A^{*\intercal}$. Then by Fact 3.6, (13) holds.

Now using (13)(12) and a result by J. Borwein (see [9, Theorem 1] or [27, Theorem 3.1.4(i)]), we have $k = k^{**} = (k^* + \iota_{\operatorname{dom}\partial k^*})^* = (\langle\cdot,\cdot\rangle + \iota_{\operatorname{gra} A^{-1}})^* = F_A$. ■

Definition 3.8 (Fitzpatrick functions of order n) [1, Definition 2.2 and Proposition 2.3] *Let $A: X \rightrightarrows X^*$. For every $n \in \{2,3,\ldots\}$, the* Fitzpatrick function of A of order n *is*

$$F_{A,n}(x,x^*) := \sup_{\{(a_1,a_1^*),\cdots(a_{n-1},a_{n-1}^*)\}\subseteq \operatorname{gra} A} \Big(\langle x,x^*\rangle + \Big(\sum_{i=1}^{n-2}\langle a_{i+1}-a_i, a_i^*\rangle\Big) + \langle x-a_{n-1}, a_{n-1}^*\rangle + \langle a_1 - x, x^*\rangle\Big).$$

Clearly, $F_{A,2} = F_A$. We set $F_{A,\infty} = \sup_{n\in\{2,3,\cdots\}} F_{A,n}$.

Fact 3.9 (recursion) (See [4, Proposition 2.13].) *Let $A : X \rightrightarrows X^*$ be monotone, and let $n \in \{2, 3, \ldots\}$. Then*

$$F_{A,\,n+1}(x, x^*) = \sup_{(a,a^*)\in \operatorname{gra} A} \big(F_{A,\,n}(a, x^*) + \langle x - a, a^*\rangle\big), \quad \forall (x, x^*) \in X \times X^*.$$

Theorem 3.10 *Let $A : X \rightrightarrows X^*$ be a maximal monotone and symmetric linear relation, let $n \in \{2, 3, \ldots\}$, and let $(x, x^*) \in X \times X^*$. Then*

$$F_{A,\,n}(x, x^*) = \tfrac{n-1}{n}\overline{q_A}(x) + \tfrac{n-1}{n}q_A^*(x^*) + \tfrac{1}{n}\langle x, x^*\rangle, \tag{14}$$

consequently, $F_{A,\,n}(x, x^*) = \frac{2(n-1)}{n}F_A(x, x^*) + \frac{2-n}{n}\langle x, x^*\rangle$. Moreover,

$$F_{A,\,\infty} = \overline{q_A} \oplus q_A^* = 2F_A - \langle \cdot, \cdot\rangle. \tag{15}$$

Proof. Let $(x, x^*) \in X \times X^*$. The proof is by induction on n. If $n = 2$, then the result follows for Proposition 3.7.

Now assume that (14) holds for $n \geq 2$. Using Fact 3.9, we see that

$$\begin{aligned}
&F_{A,\,n+1}(x, x^*) = \sup_{(a,a^*)\in \operatorname{gra} A} \Big(F_{A,\,n}(a, x^*) + \langle x - a, a^*\rangle\Big)\\
&= \sup_{(a,a^*)\in \operatorname{gra} A} \Big(\tfrac{n-1}{n}q_A^*(x^*) + \tfrac{n-1}{n}\overline{q_A}(a) + \tfrac{1}{n}\langle a, x^*\rangle + \langle x - a, a^*\rangle\Big)\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \sup_{(a,a^*)\in \operatorname{gra} A} \Big(\tfrac{n-1}{2n}\langle a, a^*\rangle + \langle a, \tfrac{1}{n}x^*\rangle + \langle x, a^*\rangle - \langle a, a^*\rangle\Big), \quad \text{(by Proposition 3.3(i))}\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \sup_{(a,a^*)\in \operatorname{gra} A} \Big(\langle a, \tfrac{1}{n}x^*\rangle + \langle x, a^*\rangle - \tfrac{n+1}{2n}\langle a, a^*\rangle\Big)\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \tfrac{2n}{n+1} \sup_{(a,a^*)\in \operatorname{gra} A} \Big(\langle \tfrac{n+1}{2n}a, \tfrac{1}{n}x^*\rangle + \langle x, \tfrac{n+1}{2n}a^*\rangle - \langle \tfrac{n+1}{2n}a, \tfrac{n+1}{2n}a^*\rangle\Big)\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \tfrac{2n}{n+1} \sup_{(b,b^*)\in \operatorname{gra} A} \Big(\langle b, \tfrac{1}{n}x^*\rangle + \langle x, b^*\rangle - \langle b, b^*\rangle\Big)\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \tfrac{2n}{n+1}F_A(x, \tfrac{1}{n}x^*)\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \tfrac{n}{n+1}q_A^*(\tfrac{1}{n}x^*) + \tfrac{n}{n+1}\overline{q_A}(x) + \tfrac{1}{n+1}\langle x^*, x\rangle \quad \text{(by Proposition 3.7)}\\
&= \tfrac{n-1}{n}q_A^*(x^*) + \tfrac{1}{(n+1)n}q_A^*(x^*) + \tfrac{n}{n+1}\overline{q_A}(x) + \tfrac{1}{n+1}\langle x^*, x\rangle\\
&= \tfrac{n}{n+1}q_A^*(x^*) + \tfrac{n}{n+1}\overline{q_A}(x) + \tfrac{1}{n+1}\langle x, x^*\rangle,
\end{aligned}$$

which is the result for $n+1$. Thus, by Proposition 3.7, $F_{A,\,n}(x, x^*) = \frac{2(n-1)}{n}F_A(x, x^*) + \frac{2-n}{n}\langle x, x^*\rangle$. By (14), $\operatorname{dom} F_{A,\,n} = \operatorname{dom}(\overline{q_A} \oplus q_A^*)$. Now suppose that $(x, x^*) \in \operatorname{dom} F_{A,\,n}$.
By $\overline{q_A}(x) + q_A^*(x^*) - F_{A,\,n}(x, x^*) = \frac{1}{n}\Big(\overline{q_A}(x) + q_A^*(x^*) - \langle x, x^*\rangle\Big) \geq 0$ and

$$F_{A,\,n}(x, x^*) \to (\overline{q_A} \oplus q_A^*)(x, x^*),\ n \to \infty.$$

Thus, (15) holds. ■

Remark 3.11 Theorem 3.10 generalizes and simplifies [1, Example 4.4] and [3, Example 6.4]. See Corollary 3.13.

Remark 3.12 Formula Identity (14) does not hold for nonsymmetric linear relations. See [3, Example 2.8] for an example when A is skew linear operator and (14) fails.

Corollary 3.13 *Let $A : X \to X^*$ be a maximal monotone and symmetric linear operator, let $n \in \{2, 3, \ldots\}$, and let $(x, x^*) \in X \times X^*$. Then*

$$F_{A,n}(x, x^*) = \tfrac{n-1}{n} q_A(x) + \tfrac{n-1}{n} q_A^*(x^*) + \tfrac{1}{n}\langle x, x^*\rangle, \tag{16}$$

and,

$$F_{A,\infty} = q_A \oplus q_A^*. \tag{17}$$

If X is a Hilbert space, then

$$F_{\mathrm{Id},n}(x, x^*) = \tfrac{n-1}{2n}\|x\|^2 + \tfrac{n-1}{2n}\|x^*\|^2 + \tfrac{1}{n}\langle x, x^*\rangle, \tag{18}$$

and,

$$F_{\mathrm{Id},\infty} = \tfrac{1}{2}\|\cdot\|^2 \oplus \tfrac{1}{2}\|\cdot\|^2. \tag{19}$$

Definition 3.14 *Let $F_1, F_2 \colon X \times X^* \to \left]-\infty, +\infty\right]$. Then the* partial inf-convolution $F_1 \Box_2 F_2$ *is the function defined on $X \times X^*$ by*

$$F_1 \Box_2 F_2 \colon (x, x^*) \mapsto \inf_{y^* \in X^*} \big(F_1(x, x^* - y^*) + F_2(x, y^*)\big).$$

Theorem 3.15 (nth order Fitzpatrick function of the sum) *Let $A, B \colon X \rightrightarrows X^*$ be maximal monotone and symmetric linear relations, and let $n \in \{2, 3, \cdots\}$. Suppose that* $\operatorname{dom} A - \operatorname{dom} B$ *is closed. Then $F_{A+B,n} = F_{A,n} \Box_2 F_{B,n}$. Moreover, $F_{A+B,\infty} = F_{A,\infty} \Box_2 F_{B,\infty}$.*

Proof. By [23, Theorem 5.5] or [25], $A + B$ is maximal monotone. Hence $A + B$ is a maximal monotone and symmetric linear relation. Let $(x, x^*) \in X \times X^*$. Then by Theorem 3.10,

$$\begin{aligned}
&F_{A,n} \Box_2 F_{B,n}(x, x^*)\\
&= \inf_{y^* \in X^*} \Big(\tfrac{2(n-1)}{n} F_A(x, y^*) + \tfrac{2-n}{n}\langle x, y^*\rangle + \tfrac{2(n-1)}{n} F_B(x, x^* - y^*) + \tfrac{2-n}{n}\langle x, x^* - y^*\rangle\Big)\\
&= \tfrac{2-n}{n}\langle x, x^*\rangle + \inf_{y^* \in X^*} \tfrac{2(n-1)}{n}\Big(F_A(x, y^*) + F_B(x, x^* - y^*)\Big)\\
&= \tfrac{2-n}{n}\langle x, x^*\rangle + \tfrac{2(n-1)}{n} F_A \Box_2 F_B(x, x^*)\\
&= \tfrac{2-n}{n}\langle x, x^*\rangle + \tfrac{2(n-1)}{n} F_{A+B}(x, x^*), \quad \text{(by [7, Theorem 5.10])}\\
&= F_{A+B,n}(x, x^*) \quad \text{(by Theorem 3.10)}.
\end{aligned}$$

Similarly, using (15), we have $F_{A+B,\infty} = F_{A,\infty} \Box_2 F_{B,\infty}$. ■

Remark 3.16 Theorem 3.15 generalizes [3, Theorem 5.4].

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